On the existence of Killing fields in smooth spacetimes with a compact Cauchy horizon

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Abstract

We prove that the surface gravity of a compact non-degenerate Cauchy horizon in a smooth vacuum spacetime, can be normalized to a non-zero constant. This result, combined with a recent result by Oliver Petersen and István Rácz, end up proving the Isenberg-Moncrief conjecture on the existence of Killing fields, in the smooth differentiability class. The well known corollary of this, in accordance with the strong cosmic censorship conjecture, is that the presence of compact Cauchy horizons is a non-generic phenomenon. Though we work in $3+1$, the result is valid line by line in any $n+1$-dimensions.

1 Introduction

This article discusses the existence of Killing fields on *smooth* (i.e. C^∞) time-orientable smooth vacuum $3+1$ - spacetimes $(\mathcal{M};g)$ having a compact connected Cauchy horizon $\mathcal{C}$. To make the setup clear from the start, the horizon $\mathcal{C}$ is assumed to divide $\mathcal{M}$ into two connected regions, one of which, $\mathcal{H}$, is a globally hyperbolic spacetime having a smooth closed three-manifold as a Cauchy surface. Such $\mathcal{C}$ is known to be always a smooth [4], [6], [5], totally geodesic null hypersurface of $\mathcal{M}$, ruled by null geodesics [16]. We will assume throughout that $\mathcal{C}$ is *non-degenerate*, namely, that there is on it at least one future or past incomplete null geodesic, (recall that an inextensible geodesic is *incomplete* if it has finite affine length). The "future" direction is relabeled if necessary so that at least an incomplete null geodesic points into it.

Cauchy horizons are rather unique and peculiar objects in the General theory of Relativity whose conceptual and theoretical significance can be hardly overlooked. Spacetimes having Cauchy horizons contain regions that, in a general sense, cannot be predicted from the initial data over the Cauchy surface and therefore display properties that typically conflict our intuition and the causal foundation of classical physics. This behavior is explicit in well known examples, as in the Taub-NUT family [7], where some vacuum solutions with the same initial data share their globally hyperbolic bulks but differ (i.e. are non-isometric) beyond a Cauchy horizon. In other words, some globally hyperbolic spacetimes can be extended beyond a Cauchy horizon in several inequivalent ways. In this sense, causality beyond the horizon is lost. The literature about compact Cauchy horizons is extense. For further examples, discussions and viewpoints we refer the reader to [1], [15], [8] and references therein. Despite of all that, according to the strong cosmic censorship conjecture, Cauchy horizons are objects that should not exist generically, more specifically: generic[(1)] smooth vacuum initial data should give rise to unique *inextensible* maximal globally hyperbolic spacetimes, hence without any kind of horizon, (see for instance Geroch-Horowitz in [2]).

In 1983 James Isenberg and Vincent Moncrief started a series of seminal investigations [9], [3], [10], [11] to demonstrate that smooth vacuum spacetimes with compact Cauchy horizons, as those of the Taub-NUT vacuum solutions, always have a Killing field [9]. As spacetimes with Killing fields are non-generic, a proof of the Isenberg-Moncrief conjecture, (as we will call it from now on), would imply that (at least) compact Cauchy horizons are non-generic objects either. Though still far from strong cosmic censorship (because Cauchy horizons could be non-compact), a proof of the Isenberg-Moncrief conjecture would provide further support to it.

Despite of the significant progress achieved in the series of works [9], [3], [10], and [11], that showed in particular the existence of Killing fields for analytic vacuum spacetimes with non-ergodic horizons[(2)], the general proof of the conjecture for smooth spacetimes has been so far elusive. In this article we conclude the Isenberg-Moncrief program by proving a missing technical point that together with recent breakthroughs by Petersen and Rácz in [13] and by Petersen in [12] end up proving the conjecture in the smooth differentiability class of vacuum spacetimes. We discuss all that in the following lines.

As in the beginning of the introduction, assume that $(M; g)$ is a time-oriented spacetime with a compact Cauchy horizon $\mathcal{C}$, (hence, as said, $\mathcal{C}$ is a smooth null and totally geodesic hypersurface). Assume for the moment too that that $\mathcal{C}$ is a Killing horizon, that is, there exists a Killing field K on $\mathcal{H}$, that when restricted to $\mathcal{C}$ is null, non-zero and tangent to it. Under these hypotheses it is well known that $\nabla_K K = \kappa K$ where κ is a constant known as the surface gravity (see [16], Ch. 12.5.). If $\kappa \neq 0$ then the horizon is non-degenerate (see Proposition 2 below) and one can scale K if necessary to have,

$$\nabla_K K = -K. \tag{1.1}$$

Thus, a necessary condition for $\mathcal{C}$ to be a non-degenerate Killing horizon is the existence of a smooth null vector field K over $\mathcal{C}$ satisfying (1.1). What is remarkable is that this necessary property is also sufficient. For analytic spacetimes this fact was proved by Isenberg and Moncrief already in [9] (it must be assumed that K is analytic too). For smooth spacetimes instead it is the result of a new breakthrough by Petersen-Rácz in [13]. Furthermore, also for smooth spacetimes, Petersen in [12] has shown that K also extends to a smooth Killing field on a neighbourhood of $\mathcal{C}$ in the complement of $\mathcal{H}$. Thus, if $(M; g)$ is smooth, K extends to a smooth Killing field on both sides of $\mathcal{C}$. We summarize these last two results in the following Theorem.

Theorem 1 (Petersen-Rácz [13], Thm. 1.2; Petersen [12], Thm. 1.4.)**.** *Let $(M; g)$ be a smooth time-orientable vacuum spacetime with a compact connected Cauchy horizon $\mathcal{C}$. Suppose that there exists a smooth null vector field K on $\mathcal{C}$ such that $\nabla_K K = -K$. Then, K extends to a smooth Killing vector field K all over the globally hyperbolic region $\mathcal{H}$, and at least to a neighbourhood of the horizon, in the complement of $\mathcal{H}$.*

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(1) Generic here means "on an open and dense set" of initial data.

(2) For the notion of ergodic horizon, that won't play any role here, see [9]

This article is devoted exclusively to proving the existence of such smooth vector field K over $\mathcal{C}$, tangent to the null generators and satisfying $\nabla_K K = -K$. After proving that, the Isenberg-Moncrief conjecture in the smooth class of vacuum spacetimes follows as a corollary to Theorem 1. For analytic spacetimes, Isenberg and Moncrief succeeded in proving the existence of an analytic K in all possible scenarios they considered (depending on the orbital type of the null generators), except when the horizon is ergodic (see [9]). In all instances, the construction relied on a suitable implementation of a so called "ribbon" argument. Our construction here also follows a ribbon argument, however with certain new ingredients that we explain later. Contrary to the Isenberg and Moncrief constructions of K, the analysis here is general and does not depend on the orbital structure of the null generators.

Sometimes, the existence of K is phrased as saying that "the surface gravity can be normalized to a non-zero constant" (see Def. 1.1 in [13]). We used this expression inside the abstract.

We proceed now to explain below how to carry over the task of finding K. We begin by establishing an equivalence that will turn out to be rather useful.

Proposition 2. *If a non-necessarily smooth, null and nowhere zero vector field $\tilde{K}$ on $\mathcal{C}$ satisfies*

$$\nabla_{\tilde{K}} \tilde{K} = -\tilde{K}, \tag{1.2}$$

then every future null geodesic is incomplete and for every $p \in \mathcal{C}$, $\tilde{K}(p)$ is the only future null vector at p such that the null geodesic starting at p with velocity $\tilde{K}(p)$, has affine length equal to one.

Conversely, if all future null geodesics are incomplete, and if for every p in $\mathcal{C}$ we define $\tilde{K}(p)$ as the only future null vector such that the null geodesic starting at p with velocity $\tilde{K}(p)$ has finite affine length one, the the resulting vector field $\tilde{K}$ satisfies (1.2).

Note that if a non-necessarily smooth nowhere zero and null $\tilde{K}$ satisfies (1.2) then $\tilde{K}$ is necessarily smooth (i.e. C^∞) along the null generators, though of course not necessarily smooth along the directions transversal to them. Let us see now the proof of this equivalence.

Let $\tilde{K}$ be a non-zero vector field on $\mathcal{C}$, tangent to the null generators and satisfying (1.2). Let p be any point on $\mathcal{C}$ and let $\gamma(s)$ be the null geodesic with $\gamma(0) = p$ and $\gamma'(0) = \tilde{K}(p)$. Then $\gamma'(s) = f(s)\tilde{K}(\gamma(s))$ for some smooth $f(s)$. Thus (1.2) implies $f' - f^2 = 0$ and $\gamma'(0) = \tilde{K}(p)$ implies $f(0) = 1$. Hence $f(s) = 1/(1-s)$, proving that the affine length of γ must be one. Then observe that if the geodesic $\gamma(s)$ were to start instead with a different velocity $\lambda\tilde{K}(p)$, with $\lambda > 0$ and $\lambda \neq 1$, then the affine length would be $1/\lambda$, hence different from one. We have then shown the direct implication of the proposition. We prove now the converse. So suppose that all the future pointing null geodesics on $\mathcal{C}$ are incomplete. Then, for any point p on $\mathcal{C}$ let $\tilde{K}(p)$ be the only future null vector at p such that the affine length of the inextensible null geodesic $\gamma(s)$ starting from p with velocity $\tilde{K}(p)$ is equal to one. We will prove in what follows that then (1.2) must hold, that is $\nabla_{\tilde{K}}\tilde{K} = -\tilde{K}$. Again, let $\gamma(s) : [0, 1) \to \mathcal{C}$ be the future null geodesic on $\mathcal{C}$ such that $\gamma(0) = p$ and $\gamma'(0) = \tilde{K}(p)$. We claim that along $\gamma(s)$ the vector field $\tilde{K}$ must adopt the expression,

$$\tilde{K}(\gamma(s)) = (1-s)\gamma'(s). \tag{1.3}$$

Indeed, for any $0 \leq s_0 < 1$, the geodesic starting from $\gamma(s_0)$ with velocity $\gamma'(s_0)$ has obviously affine length equal to $1 - s_0$. Hence, the geodesic starting from $\gamma(s_0)$ with velocity $(1 - s_0)\gamma'(s_0)$ has affine length equal to one and (1.3) follows. We can now compute,

$$\nabla_{\tilde{K}(\gamma(s_0))}\tilde{K} = (1-s_0)\nabla_{\gamma'(s_0)}((1-s)\gamma'(s))\big|_{s=s_0} = -(1-s_0)\gamma'(s_0) = -\tilde{K}(\gamma(s_0)), \tag{1.4}$$

where we have used that $\nabla_{\gamma'}\gamma' = 0$ because $\gamma(s)$ is a geodesic and obviously $\nabla_{\gamma'} s = 1$. This finishes the converse and therefore the proof of the proposition.

Proposition 2 suggests that, to show the existence of a smooth null and nowhere zero vector field K on $\mathcal{C}$ satisfying (1.1), one could proceed as follows. First prove that all future geodesics are incomplete. Then prove that the vector field $\tilde{K}$, defined at every p as the only future null vector such that the null geodesic starting at p with velocity $\tilde{K}(p)$ has affine length equal to one, is indeed smooth. By Proposition 2, such vector field $\tilde{K}$ will be the smooth field K that we are looking for. This is the strategy that we will follow. The vector field $\tilde{K}$ will be called the *candidate vector field* (candidate to be the vector field K).

Based on the discussion above, the Isenberg-Moncrief conjecture in the smooth class will follow as a corollary of the next theorem (that will be our main theorem) and Theorem 1.

Theorem 3 (Main Theorem)**.** *Let $\mathcal{C}$ be a compact connected and non-degenerate Cauchy horizon on a time orientable smooth spacetime. Then, all the future null geodesics of $\mathcal{C}$ have finite affine length. Furthermore, the candidate vector field is smooth.*

Let Z be a smooth future null and nowhere zero vector field defined on an open set $U \subset \mathcal{C}$ of a local chart $\psi^{-1} : U \to \mathbb{R}^3$. Let $L(x, y, z)$ be the affine length of the null geodesic starting at $\psi(x, y, z)$ with velocity $Z(\psi(x, y, z))$. If $L < \infty$, then the candidate vector field $\tilde{K}$ adopts the local presentation,

$$\tilde{K}(\psi(x, y, z)) = L(x, y, z)Z(\psi(x, y, z)), \tag{1.5}$$

Hence, if $L(x, y, z)$ is furthermore smooth, then $\tilde{K}$ will be smooth on U. This local expression will be useful.

In basic terms, the ribbon argument exploits the following crucial fact: For any null and nowhere zero vector field Z tangent to $\mathcal{C}$ and defined on an open set U of $\mathcal{C}$, the one-form ω_Z on U defined by,

$$\nabla_Y Z =: \omega_Z(Y)Z, \tag{1.6}$$

is *null-closed*, that is,

$$d\omega_Z(Z, Y) = 0, \tag{1.7}$$

for any Y tangent to $\mathcal{C}$. This fact is due to the vacuum Einstein equations and a proof of it in Gaussian null coordinates can be found in section E of [11], (coordinate independent proofs can be given too, [(3)]).

Suppose now that a global smooth future null and nowhere zero vector field Z on C is given and fixed. Let $\varphi_p(z)$ be the integral curve of Z, starting at p when $z = 0$ ($\varphi'_p(z) = Z(\varphi_p(z))$. The orbits $\varphi_p(z)$ are of course reparametrized future null geodesics. Then, a direct (standard) computation, that we explain in section 3, gives,

$$L(p) = \int_0^\infty e^{\int_0^\rho \omega_Z(\varphi'_p(z))dz} d\rho, \tag{1.8}$$

for the affine length $L(p)$ of the null geodesic starting at p with velocity $Z(p)$. This affine length can be a priori finite or infinite. Observe that this expression doesn't depend on which vector field Z is used. Now, suppose that $\alpha(\lambda)$, $\lambda \in [0,1]$ is a smooth curve transversal to the null generators, and joining a point p_0 to a point p. Then, for any $\rho > 0$ the map,

$$\Psi : [0,1] \times [0,\rho] \to \mathcal{C}, \tag{1.9}$$

given by $\Psi(\lambda, z) = \varphi_{\alpha(\lambda)}(z)$, is a smooth immersion called the ribbon. In simple words, the ribbon results after translating the graph of α by the flow of Z. This surface is of course ruled by null generators because Z is null. This fact and (1.7), show that the integral of $d\omega_Z$ on the surface is zero[(4)]. Therefore, by Stoke's theorem, the integral of ω_Z along the boundary of the surface is zero too. We write this in the following convenient form,

$$\int_0^\rho \omega_Z(\varphi'_p(z))dz = S(\rho) - S(0) + \int_0^\rho \omega_Z(\varphi'_{p_0}(z))dz, \tag{1.10}$$

where $S(\rho)$ is the integral of ω_Z along the translation of α by the flow of Z, i.e. the curve $\lambda \to \varphi_{\alpha(\lambda)}(\rho)$. When this expression is inserted inside the exponent in the integrand of (1.8) we obtain,

$$L(p) = \int_0^\infty e^{S(\rho)-S(0)} e^{\int_0^\rho \omega_Z(\varphi'_{p_0}(z))dz} d\rho. \tag{1.11}$$

As $L(p_0)$ is given by,

$$L(p_0) = \int_0^\infty e^{\int_0^\rho \omega_Z(\varphi'_{p_0}(z))dz} d\rho, \tag{1.12}$$

these last two equations somehow represent a way of linking $L(p)$ to $L(p_0)$. In particular, if it is known that $L(p_0) < \infty$, (i.e. if the future null geodesic from p_0 is incomplete), and $S(\rho)$ is bounded for all $\rho > 0$, then $L(p) < 0$. This could be a way of proving the required first step that all future null geodesics are incomplete, out of the information that the horizon is non-degenerate, namely that there is at least one incomplete future null geodesic. The problem here is that, if the curve α is translated by any globally defined Z, then, unless some extra information comes into play, there is little chance to prove boundedness of S because the curve $\lambda \to \varphi_{\alpha(\lambda)}(\rho)$ could get increasingly distorted as $\rho \to \infty$. Proving the required step that all future null geodesics are incomplete (i.e. proving that $L < \infty$) using this implementation of the ribbon argument, is not viable and something different must be done. It is somehow intuitive that the key is to work with ribbons where the "top" and "bottom" sides (curves) are a priori controlled.

[(3)] We are indebted to Oliver Petersen for showing us an unpublished intrinsic calculation.

[(4)] Actually the integral of the pull-back of $d\omega_Z$ to $[0,1] \times [0,\rho]$

To tackle this difficulty, in this article we introduce and make use of the notion of horizontal geodesic and horizontal parallel transport that depend upon fixing any smooth global distribution of two-planes transversal to the null directions (see Section 2). Then, briefly (see full details in the body of the article), horizontal geodesics are curves whose velocity field is horizontally parallel and that is tangent to the two-planes of the distribution. Now, fix any null, future, and nowhere zero smooth vector field X on $\mathcal{C}$. Let α be a horizontal geodesic starting at a point p_0 and ending at a point p. This horizontal geodesic is tangent to the distribution of two-planes and is thus transversal to the null generators. One can then "horizontally transport" α along the integral curve of X starting at p_0. The key point of this process, proven in Proposition 6, is that the orbit described by any point in α that we pick, is always a null curve with non-zero velocity. Hence, the result of horizontally translating α is a ribbon whose "top" and "bottom" are two horizontal geodesics, hence having controlled geometry. The vector field Z used for the ribbon argument, is (essentially) the velocity field obtained while transporting α. To end, and without entering into details that can easily be found in the body of the text, the result of implementing the ribbon argument in this way is a local presentation of the candidate vector field, as in (1.5), from which one can easily prove the incompleteness of the future null geodesics (i.e. prove $L < \infty$) and the smoothness of the candidate vector field (i.e. prove the smoothness of L).

The notions of "horizontal geometry", including the notion of horizontal geodesic and horizontal exponential map are discussed in Section 2. The main theorem is treated in the Section 3 and in Section 4 we prove the couple of propositions stated in Section 2, including Proposition 6.

2 The horizontal exponential map

Let $\mathfrak{p} : T\mathcal{C} \to \mathcal{C}$ be the tangent bundle of $\mathcal{C}$, (points in $T\mathcal{C}$ are denoted as usual by (p, v), with $\mathfrak{p}(p, v) = p$). Let $\mathfrak{p} : N \to \mathcal{C}$ be the vector-bundle of null vectors tangent to $\mathcal{C}$. We call N the *null bundle*. Let $\mathfrak{p} : H \to \mathcal{C}$ be any smooth distribution of two-planes in $T\mathcal{C}$, that we think as a vector bundle with two-dimensional fibers, such that $T\mathcal{C} = N \oplus H$. Having chosen H, we call it the *horizontal bundle*. The fibers of N and H over p will be denoted by $N(p)$ and $H(p)$ respectively. Let $\pi : T\mathcal{C} = N \oplus H \to H$ be the natural projection (i.e. if $u = v \oplus w$ with $u \in T_p\mathcal{C}$, $v \in N(p)$ and $w \in H(p)$ then $\pi(p, u) = (p, w)$).

A smooth vector field Y on $\mathcal{C}$ is said to be *horizontal* if $Y(p) \in H(p)$ for all $p \in \mathcal{C}$ (i.e. Y is a smooth section of H). Let ∇ be the space-time covariant derivative restricted to $\mathcal{C}$ (recall that $\mathcal{C}$ is totally geodesic). Define the *horizontal covariant derivative* D on H as follows: if X is a vector field on $\mathcal{C}$ and Y is a horizontal vector field, then,

$$D_X Y := \pi(\nabla_X Y). \tag{2.1}$$

This horizontal covariant derivative defines *horizontal parallel fields* over curves in the usual manner: a horizontal vector field $V : (a, b) \to H$ over a curve $\gamma : (a, b) \to \mathcal{C}$ (that is $V(s) \in H(\gamma(s))$ for all $s \in (a, b)$), is *parallel* iff $D_{\gamma'} V = 0$. Given $\gamma : [a, b] \to \mathcal{C}$ and a vector $v \in H(\gamma(a))$ one can always *parallel transport* v along γ obtaining thus a horizontal parallel field V over γ with $V(a) = v$. Observe that D is compatible with the spacetime metric g restricted to H, namely if Y and Z are horizontal vector fields, and X is a vector field on $\mathcal{C}$ then,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\pi(\nabla_X Y, Z) + g(Y, \pi(\nabla_X Z)) \tag{2.2}$$
$$= g(D_X Y, Z) + g(Y, D_X Z). \tag{2.3}$$

We assume from now on that H is endowed with the metric g.

A curve $\gamma : (a,b) \to \mathcal{C}$ is said to be a *horizontal geodesic* if $\gamma'(s) \in H(\gamma(s))$ for all $s \in (a,b)$ and $D_{\gamma'}\gamma' = 0$. The following basic proposition on the existence of horizontal geodesics will be proved in Section 4.

Proposition 4 (Existence and uniqueness)**.** *Given $p \in \mathcal{C}$ and $v \in H(p)$, there is a unique horizontal geodesic $\gamma : (-\infty,\infty) \to \mathcal{C}$ with $\gamma(0) = p$ and $\gamma'(0) = v$.*

Then we define the horizontal exponential map, in the usual manner.

Definition 5. *The horizontal exponential map, is the map $\overline{\exp} : H \to \mathcal{C}$ defined as,*

$$\overline{\exp}\,(p,v) = \gamma(1), \tag{2.4}$$

where $\gamma(s)$ is the unique horizontal geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$.

The map $\overline{\exp}$ will be of course smooth (it comes after solving a smooth ODE). The next proposition states the only crucial (unsubstitutable) property of the horizontal exponential map that we will need during the proof of the main theorem. Before it, define a curve $\alpha : [a,b] \to \mathcal{C}$ to be *null* if $\alpha'(s) \in N(\alpha(s))$ for all $s \in [a,b]$.

Proposition 6 (Transport of horizontal geodesics)**.** *Let $p \in \mathcal{C}$ and $v \in H(p)$, $v \neq 0$. Let $\alpha : [a,b] \to \mathcal{C}$ be a null curve with nowhere zero velocity and $\alpha(a) = p$, and let $V : [a,b] \to \mathcal{C}$ be the parallel transport of v along α. Then the curve $\beta : [a,b] \to \mathcal{C}$ given by $\beta(s) = \overline{\exp}\,(\alpha(s), V(s))$ is a null curve with nowhere zero velocity.*

We prove this proposition also in Section 4.

3 Proof of Theorem 3

Before going into the proof we mention two important preliminary facts that will be used.

(I) First, and as mentioned in the introduction, any null vector field Z on an open set U of $\mathcal{C}$ gives rise to a one-form ω_Z over U defined by,

$$\omega_Z(Y)Z = \nabla_Y Z, \tag{3.1}$$

for any Y vector field on U. The form ω_Z has the central property that its exterior derivative is *null* in the sense that,

$$d\omega_Z(Z,Y) = 0, \tag{3.2}$$

for any Y vector field on U. As commented earlier, this is the crucial property allowing the Isenberg-Moncrief “ribbon” argument and will be used fundamentally.

Note finally that if, $Z = fX$ then,

$$\omega_Z(Z) = \frac{Z(f)}{f} + \omega_X(Z). \tag{3.3}$$

(II) Second, we mention how to compute the affine length of a null geodesic from a non-affine parametrization of it. Let $\gamma(s)$, $\gamma : [0, L) \to \mathcal{C}$ be an inextensible null geodesic. The affine length L being finite or infinite. Let $s(\rho) : [\rho_0, \infty) \to [0, L)$ be a smooth change of parameter so that now $\gamma(s(\rho))$ is a null geodesic possibly with a non-affine parameterization. Letting $\gamma' = d\gamma/d\rho$ and $s' = ds/d\rho$, we compute,

$$\nabla_{\gamma'}\gamma' = (\frac{s''}{s'})\gamma' =: \omega(\rho)\gamma', \tag{3.4}$$

Hence, having the expression for $\omega(\rho)$, the affine-length L is computed by,

$$L = s'(0)\int_{\rho_0}^{\infty} e^{\int_{\rho_0}^{\rho}\omega(\lambda)d\lambda} d\rho. \tag{3.5}$$

We deduce then, that in order to have affine length L equal to one, it is necessary and sufficient to have,

$$\frac{1}{s'(0)} = \int_{\rho_0}^{\infty} e^{\int_{\rho_0}^{\rho}\omega(\lambda)d\lambda} d\rho < \infty. \tag{3.6}$$

This is equivalent to have $\int_{\rho_0}^{\infty} e^{\int_{\rho_0}^{\rho}\omega(\lambda)d\lambda} d\rho < \infty$ and to start the geodesic $\gamma(s)$ at $\gamma(0)$ with velocity equal to,

$$\frac{d\gamma}{ds}\bigg|_{s=0} = (\int_{\rho_0}^{\infty} e^{\int_{\rho_0}^{\rho}\omega(\lambda)d\lambda} d\rho)\frac{d\gamma}{d\rho}\bigg|_{\rho=\rho_0}. \tag{3.7}$$

We will use this expression to given an explicit (local) presentation of the candidate vector field, that will be proved to be smooth.

Finally, note that if X is a nowhere zero null vector field and $\gamma'(\rho) = f(\rho)X(\gamma(\rho))$ for some smooth $f(\rho) > 0$, then,

$$\omega = \frac{f'}{f} + \omega_X(\gamma'), \tag{3.8}$$

(compare this with (3.3)).

We introduce now additional notation.

During the proof, $B(0, r) \subset \mathbb{R}^2$ will denote the open ball of radius $r > 0$ and centered at the origin. In $\mathbb{R}^2$ we use coordinates (x, y).

It turns out that in order to parameterize incomplete geodesics in a controlled fashion it will be convenient to fix an auxiliary smooth nowhere zero future null vector field X. We fix such X from now on and reserve the letter X for it. Let X^* be the one form such that $X^*(X) = 1$ and $X^*(Y) = 0$ for any horizontal vector field Y. The form X^* is clearly smooth. Let $\varphi : \mathcal{C} \times (-\infty, \infty) \to \mathcal{C}$ be the smooth flow defined by X, namely,

$\varphi(p,z)$ is the solution to the ODE,

$$\frac{d\varphi(p,z)}{dz} = X(\varphi(p,z)), \quad \varphi(p,0) = p, \tag{3.9}$$

hence with z the parameter of the integral curves of X.

Finally we let $\mathfrak{p} : E \to \mathcal{C}$ be the bundle of orthonormal frames of H. Points in E are denoted by $(p, \{e_1, e_2\})$ with $\{e_1, e_2\}$ an orthonormal basis (frame) of $H(p)$. Given $(p, \{e_1, e_2\})$ we denote by $\{e_1(z), e_2(z)\}$ to the horizontal parallel transport of $\{e_1, e_2\}$ along the null curve $z \to \varphi(p,z)$.

Proof of Theorem 3. We begin explaining the main arguments of the proof. The proof is divided in two obvious consecutive steps: (A) proving that all future null geodesics have finite affine length, (B) proving that the candidate Killing vector field $\tilde{K}$ is smooth. For (A) it will be enough to show that: (A') there are uniform $\epsilon > 0$ and $\delta > 0$ such that, if a future null geodesic from a point p_0 is incomplete, then all future null geodesics starting at any point in the uniform neighbourhood,

$$U(p_0, \epsilon, \delta) = \{\overline{\exp}(\varphi(p_0, z), xe_1^0(z) + ye_2^0(z)) : \ z^2 < \delta^2, \ x^2 + y^2 < \epsilon^2\}, \tag{3.10}$$

are also incomplete, (above $\{e_1^0, e_2^0\}$ is any frame in $E(p_0)$). What is important here is that ϵ and δ are independent on p_0. Indeed, if we show (A') then the set of points with a null future incomplete geodesic will be open and closed, and thus (A) will follow from the connectivity of $\mathcal{C}$. Now, to prove (A'), but also for the proof of the step (B), it will play a simple but important role the map,

$$(x, y, z) \to \overline{\exp}(\varphi(p_0, z), xe_1^0(z) + ye_2^0(z)), \tag{3.11}$$

that we used to define $U(p_0, \epsilon, \delta)$ in (3.10). Let us define it precisely in the next lines and inspect its properties. We will end up explaining the argument behind the proof of (B).

Given p_0 and $\{e_1^0, e_2^0\} \in E(p_0)$, and given $\epsilon > 0$ and $\delta > 0$, define

$$\psi : B(0, \epsilon) \times (-\delta, \infty) \subset \mathbb{R}^3 \to \mathcal{C}, \tag{3.12}$$

as,

$$\psi(x, y, z) = \overline{\exp}\,(\varphi(p_0, z), xe_1^0(z) + ye_2^0(z)). \tag{3.13}$$

Now, it is not difficult to prove and we will do later, that if $\epsilon > 0$ and $\delta > 0$ are small enough, then for any $z_1 \geq 0$ the restriction of ψ to $B(0, \epsilon) \times (z_1 - \delta, z_1 + \delta)$ is an embedding. So let us assume such ϵ and δ for the rest of the discussion. By Proposition 6, the curves,

$$z \to \psi(x, y, z), \tag{3.14}$$

are null with non-zero velocity, and hence are null geodesics with z a non-necessarily affine parameter (this is the only place where Proposition 6 is used). Therefore, their affine length can be calculated as was explained in (II). To make that explicit define $f(x, y, z) > 0$ by,

$$d_{(x,y,z)}\psi(\partial_z) =: f(x, y, z) X(\psi(x, y, z)), \tag{3.15}$$

and then define the one-form ω^* over $B(0,\epsilon)\times(-\delta,\infty)\subset\mathbb{R}^3$ by,

$$\omega^* := \frac{df}{f} + \psi^*\omega_X, \tag{3.16}$$

which is null-closed in the sense that $d\omega^*(\partial_z, -) = 0$ indeed by virtue of (I):

$$d\omega^*(\partial_z, Y) = d(d\ln f) + d\psi^*\omega_X(\partial_z, Y) \tag{3.17}$$

$$= d\omega_X(\psi_*(\partial_z), \psi_*(Y)) = 0, \tag{3.18}$$

where in the last step we used that $\psi_*(\partial_z)$ is null. (Just in passing, ω^* is the pull-back of the form ω_Z defined by the (just) local field $Z := d\psi(\partial_z)$, see (I)). Thus, using (3.8) we obtain

$$\omega = \omega^*(\partial_z), \tag{3.19}$$

(see definition of ω in (3.4)) and then using (3.5) we find that the affine length $L(x,y,z)$ of the future null geodesic starting from $\psi(x,y,z)$ with velocity $Z(\psi(z,y,z))$ takes the expression,

$$L(x,y,z) = \int_z^\infty e^{\int_z^\rho \omega(x,y,\lambda)d\lambda} d\rho. \tag{3.20}$$

With this expression at hand, the goal of (A') is to prove that, if $L(0,0,0) < \infty$ then $L(x,y,z) < \infty$ for all (x,y,z) with $x^2 + y^2 < \epsilon$ and $z^2 < \delta^2$, whereas the goal of (B) is to prove that, once (A') is done, the following presentation of the candidate vector field,

$$\tilde{K}(\psi(x,y,z)) = L(x,y,z) Z(\psi(x,y,z)), \tag{3.21}$$

is smooth as a function of the smooth local coordinates (x,y,z). Note that as we are restricting (x,y,z) to $B(0,\epsilon)\times(-\delta,\delta)$ where ψ is an embedding, the vector field $Z(\psi(x,y,z))$ is well defined and smooth over the patch $\psi(B(0,\epsilon)\times(-\delta,\delta))$. Thus, to achieve (B) we need to show that: (B') $L(x,y,z)$ is smooth. To prove (A') and (B') we need to link somehow $L(x,y,z)$ to $L(0,0,0)$. As was explained in the introduction, linking $L(x,y,z)$ to $L(0,0,0)$ is what the ribbon argument does. We explain how it works in the following lines.

As $d\omega^*$ is null (i.e. $d\omega^*(\partial_z,-) = 0$), Stokes theorem shows that the integral of ω^* over the border of the rectangle $\mathcal{R}$ in $\mathbb{R}^3$ with vertices $(0,0,z)$, $(0,0,\rho)$, (x,y,ρ) and (x,y,z), is zero (note that ∂_z is tangent to $\mathcal{R}$). We write this identity as,

$$\int_z^\rho \omega(x,y,\lambda)d\lambda = S(x,y,\rho) - S(x,y,z) + \int_z^\rho \omega(0,0,\lambda)d\lambda, \tag{3.22}$$

where $S(x,y,z)$ is the integral of ω^* along the segment from $(0,0,z)$ to (x,y,z) and $S(x,y,\rho)$ is the one from $(0,0,\rho)$ to (x,y,ρ). Then (3.22) transforms (3.20) into,

$$L(x,y,z) = \int_z^\infty e^{S(x,y,\rho)-S(x,y,z)} e^{\int_z^\rho \omega(0,0,\lambda)d\lambda} d\rho. \tag{3.23}$$

The important point here is that positive integrand $e^{\int_z^\rho \omega(0,0,\lambda)d\lambda}$ in this integral is integrable by virtue of $L(0,0,0) < \infty$. That is the desired link between $L(0,0,0)$ and $L(x,y,z)$.

Now, we claim that (A') and (B') follow after proving that the function $S(x,y,z)$

and all the partial derivatives of it of any given order are bounded all over $B(0,\epsilon)\times(-\delta,\infty)$, (the bounds may depend on the order). Note also that the function $\omega(0,0,z)=\omega^*(\partial_z)(0,0,z)=X^*(\nabla_X X)(\varphi(p_0,z))$ that appears also in (3.23) is uniformly bounded and so are all of its derivatives.

In fact, assuming that property for $S(x,y,z)$ that we will show later, then (A') follows from the bound,

$$L(x,y,z)\leq e^{2\|S\|_{L^\infty}}L(0,0,0), \tag{3.24}$$

whereas (B') follows by a simple induction in the order of the derivatives after checking that one can use Leibniz's rule for differentiation under an improper integral sign. Let us set the induction precisely. Define $\mathcal{B}$ to be the space of functions,

$$F(x,y,z):B(0,\epsilon)\times(-\delta,\infty)\to\mathbb{R}, \tag{3.25}$$

and,

$$G(x,y,z,\rho):B(0,\epsilon)\times\{(z,\rho)\in\mathbb{R}^2:\rho\geq z\geq-\delta\}\to\mathbb{R}, \tag{3.26}$$

that are bounded and have all the derivatives of a given order also bounded. Then, the induction to prove (B') is set as follows:

If for all multi-index $I=(i_1,i_2,i_3)$, with $|I|=i_1+i_2+i_3=k$, we have,

$$\frac{\partial^{|I|}L(x,y,z)}{\partial_x^{i_1}\partial_y^{i_2}\partial_z^{i_3}}=F_I(x,y,z)+\int_z^\infty G_I(x,y,z,\rho)e^{\int_z^\rho\omega(0,0,\lambda)d\lambda}d\rho, \tag{3.27}$$

for some F_I and G_I in $\mathcal{B}$, then for all multi-index $I'=(i'_1,i'_2,i'_3)$ with $|I'|=i'_1+i'_2+i'_3=k+1$, we have,

$$\frac{\partial^{|I'|}L(x,y,z)}{\partial_x^{i'_1}\partial_y^{i'_2}\partial_z^{i'_3}}=F_{I'}(x,y,z)+\int_z^\infty G_{I'}(x,y,z,\rho)e^{\int_z^\rho\omega(0,0,\lambda)d\lambda}d\rho, \tag{3.28}$$

for some $F_{I'}$ and $G_{I'}$ in $\mathcal{B}$.

Note that for $k=0$, equation (3.20) has the form (3.27) with $F_0=0$ and $G_0=e^{S(x,y,\rho)-S(x,y,z)}$. To prove the inductive step we need to calculate the derivatives carefully. First, the derivative of (3.27) with respect to x is,

$$\frac{\partial}{\partial x}\frac{\partial^{|I|}L(x,y,z)}{\partial_x^{i_1}\partial_y^{i_2}\partial_z^{i_3}}=\frac{\partial F_I(x,y,z)}{\partial x}+\int_z^\infty\frac{\partial G_I(x,y,z,\rho)}{\partial x}e^{\int_z^\rho\omega(0,0,\lambda)d\lambda}d\rho, \tag{3.29}$$

where the differentiation inside the integral is permitted by virtue of the fact that $\partial_x(G_I e^{\int_z^\rho\omega(0,0,z)d\lambda})$ is continuous but also bounded by the integrable function of ρ, $Ce^{\int_z^\rho\omega(0,0,z)d\lambda}$. This is a pretty straightforward fact[(5)], that can be found for instance in Theorem 15 in Chp 8 of [14]. A similar calculation holds for the derivative with respect to y, whereas the derivative with respect to z is directly,

$$\frac{\partial}{\partial z}\frac{\partial^{|I|}L(x,y,z)}{\partial_x^{i_1}\partial_y^{i_2}\partial_z^{i_3}}=\frac{\partial F_I(x,y,z)}{\partial z}-G_I(x,y,z,z) \tag{3.30}$$

[(5)]The precise statement is: if $f(x,t)$ and $\partial_x f(x,t)$ are continuous, $|f(x,t)|\leq g_1(t)$ and $|\partial_x f(x,t)|\leq g_2(t)$ with $\int_{t_0}^\infty g_1(\tau)d\tau<\infty$ and $\int_{t_0}^\infty g_2(\tau)d\tau<\infty$, then the function $x\to\int_{t_0}f(x,\tau)d\tau$ is differentiable and its derivative is equal to $\int_{t_0}^\infty\partial_x f(x,\tau)d\tau$.

$$+\int_z^\infty \big(\frac{\partial G_I(x,y,z,\rho)}{\partial z} - G_I(x,y,z,\rho)\omega(0,0,z)\big)e^{\int_z^\rho \omega(0,0,\lambda)d\lambda}d\rho. \quad (3.31)$$

The proof of the inductive step thus follows. This would finish the proof of (A') and (B') and hence so of (A) and (B).

We pass now to prove the claims that were left to be proved, namely (i) to show the existence of $\epsilon > 0$ and $\delta > 0$ such that $\psi : B(0,\epsilon)\times(-\delta+z_1, \delta+z_1) \to \mathcal{C}$ is an embedding for any $z_1 \geq 0$, and (ii) show that $S(x,y,z) : B(0,\epsilon)\times(-\delta,\infty) \to \mathbb{R}$ as well as any of its derivatives are bounded (again, the bounds may depend on the order of the derivative). We prove (i) first and then (ii). Both are basically the result of compactness. Before the proof we make some analysis.

We define first a smooth map ϕ from $E \times B(0,1) \times (-1,1)$ into $\mathcal{C}$, (recall E is the frame bundle of H). Points in $E\times B(0,1)\times(-1,1)$ are denoted by $((p,\{e_1,e_2\}),(x,y),z)$. The map ϕ is given by,

$$\phi((p,\{e_1,e_2\}),(x,y),z) := \overline{\exp}\,(\varphi(p,z), xe_1(z) + ye_2(z)). \quad (3.32)$$

At any point $P = ((p,\{e_1,e_2\}),(0,0),0)$ we compute,

$$d_P\phi(\partial_x) = e_1, \quad d_P\phi(\partial_y) = e_2, \quad d_P\phi(\partial_z) = X. \quad (3.33)$$

Hence, at any $P \in E$ there is $0 < \epsilon < 1$ and $0 < \delta < 1$ such that the map ϕ restricted to $\{P\}\times B(0,3\epsilon)\times(-3\delta,3\delta)$ is an embedding. By continuity there is a neighbourhood U_P such that at any $P' \in U_P$ the map ϕ restricted to $\{P'\}\times B(0,2\epsilon)\times(-2\delta,2\delta)$ is an embedding. As E is compact then there are uniform $0 < \epsilon < 1$ and $0 < \delta < 1$, such that at any $P \in E$ the map ϕ restricted to $\{P\}\times B(0,2\epsilon)\times(-2\delta,2\delta)$ is an embedding. Also, taking into account the third equation in (3.33), that we rewrite as $X^*(d_P\phi(\partial_z)) = 1$ for all $P \in E\times\{(0,0)\}\times\{0\}$, we can decrease ϵ and δ if necessary so that, in addition,

$$X^*(d\phi(\partial_z)) \geq \frac{1}{2}, \quad (3.34)$$

all over $E\times B(0,2\epsilon)\times(-2\delta,2\delta)$. We fix such ϵ and δ from now on.

Consider now the following four smooth functions from $E\times B(0,2\epsilon)\times(-2\delta,2\delta)$ into $\mathbb{R}$,

$$\ln(X^*(d\phi(\partial_z))), \quad \omega_X(d\phi(\partial_x)), \quad \omega_X(d\phi(\partial_y)), \quad \text{and} \quad \omega_X(d\phi(\partial_z)). \quad (3.35)$$

Trivially, the four of them are bounded functions when restricted to the compact set $C := E\times\overline{B(0,\epsilon)}\times[-\delta,\delta] \subset E\times B(0,2\epsilon)\times(-2\delta,2\delta)$. The same of course holds true for any partial derivative of any order in x, y, and z. We state this as follows,

$$\|\frac{\partial^{|I|}h}{\partial x^{i_1}\partial y^{i_2}\partial z^{i_3}}\|_{L^\infty(C)} \leq c(|I|), \quad (3.36)$$

where I a multi-index $I = (i_1,i_2,i_3)$, $|I| = i_1+i_2+i_3$, and h is any of the four functions (3.35). We will see now that these trivial bounds are ultimately all the necessary bounds.

We are finally are in position to prove (i) and (ii). The basic observation is that for any $z_1 \geq 0$ the map ψ restricted to $B(0,\epsilon)\times(z_1-\delta, z_1+\delta)$ is "equal" to the map ϕ

restricted to $\{(\varphi_{z_1}(p_0), \{e_1^0(z_1), e_2^0(z_1)\})\} \times B(0,\epsilon) \times (-\delta, \delta)$, more precisely if we define,

$$\chi : B(0,\epsilon) \times (-\delta + z_1, \delta + z_1) \to \{(\varphi_{z_1}(p_0), \{e_1^0(z_1), e_2^0(z_1)\})\} \times B(0,\epsilon) \times (-\delta, \delta), \quad (3.37)$$

by,

$$\chi(x, y, z) = \phi((\varphi_{z_1}(p_0), \{e_1^0(z_1), e_2^0(z_1)\}), (x, y), z - z_1), \quad (3.38)$$

then, $\psi(x, y, z) = \phi(\chi(x, y, z))$. This shows (i), namely, that for any $z_1 \geq 0$, the map ψ restricted to $B(0,\epsilon) \times (z_1 - \delta, z_1 + \delta)$ is an embedding. To show (ii) we proceed as follows. First, as $d\chi(\partial_x) = \partial_x$, $d\chi(\partial_y) = \partial_y$ and $d\chi(\partial_z) = \partial_z$, we deduce that,

$$\ln(X^*(d\psi(\partial_z)))\Big|_{(x,y,z)} = \ln(X^*(d\phi(\partial_z)))\Big|_{\chi(x,y,z)}, \quad (3.39)$$

$$\omega_X(d\psi(\partial_x))\Big|_{(x,y,z)} = \omega_X(d\phi(\partial_x))\Big|_{\chi(x,y,z)}, \quad (3.40)$$

$$\omega_X(d\psi(\partial_y))\Big|_{(x,y,z)} = \omega_X(d\phi(\partial_y))\Big|_{\chi(x,y,z)}, \quad (3.41)$$

$$\omega_X(d\psi(\partial_z))\Big|_{(x,y,z)} = \omega_X(d\phi(\partial_z))\Big|_{\chi(x,y,z)}. \quad (3.42)$$

It follows then from this and from (3.36) that,

$$\|\frac{\partial^{|I|}\bar{h}}{\partial x^{i_1}\partial y^{i_2}\partial z^{i_3}}\|_{L^\infty(B(0,\epsilon)\times(-\delta+z_1,\delta+z_1))} \leq c(|I|), \quad (3.43)$$

where I is the multi-index $I = (i_1, i_2, i_3)$, $|I| = i_1 + i_2 + i_3$, $c(|I|)$ are the same constants as in (3.36) and $\bar{h}$ is now any of the four functions,

$$f := \ln(X^*(d\psi(\partial_z))), \quad (3.44)$$

$$\varpi_x := \omega_X(d\psi(\partial_x)), \quad \varpi_y := \omega_X(d\psi(\partial_y)), \quad \varpi_y := \omega_X(d\psi(\partial_z)). \quad (3.45)$$

Finally, as these estimates are valid for any $z_1 \geq 0$, we obtain that the form,

$$\omega^* = \frac{df}{f} + \psi^*\omega_X = \frac{\partial_x f}{f}dx + \frac{\partial_y f}{f}dy + \frac{\partial_z f}{f}dz + \varpi_x dx + \varpi_y dy + \varpi_z dx, \quad (3.46)$$

is bounded and has all its derivatives of any order bounded over $B(0,\epsilon) \times (-\delta, \infty) \subset \mathbb{R}^3$. This directly proves (ii) namely that $S(x, y, z)$ and all its derivatives of any order are bounded, as wished. □

4 Proof of Propositions 4 and 6

In this section we prove Propositions 4 and 6.

Recall that a curve $\gamma : (a, b) \to \mathcal{C}$ is a horizontal geodesic if for all $s \in (a, b)$,

$$\gamma'(s) \in H(\gamma(s)) \quad \text{and} \quad \pi(\nabla_{\gamma'}\gamma')(s) = 0. \quad (4.1)$$

Proposition 7. *Let $p \in \mathcal{C}$ and $v \in H(p)$. Then, for $\epsilon > 0$ sufficiently small, there exists a unique horizontal geodesic $\gamma : (-\epsilon, \epsilon) \to \mathcal{C}$ with $\gamma(0) = p$ and $\gamma'(0) = v$.*

Proof of Proposition 7. We make a local calculation. Let B be an embedded disc containing p and transversal to the null directions that are tangent to the null geodesics foliating $\mathcal{C}$. For $\delta > 0$ sufficiently small, the restriction of φ to $B \times (-\delta, \delta)$ is an embedding (φ is again the flow generated by X, the vector field that we fixed in Section 3). Let $U = \varphi(B \times (-\delta, \delta))$. The open set U is foliated by the null orbits $\{\{\varphi(p, z) : z \in (-\delta, \delta)\} : p \in B\}$. Let V be the quotient of U, and note that obviously V is diffeomorphic to B. Let $\xi : U \to V$ be the projection. Any function f on V lifts to a function f^* on U by: $f^*(p) := f(\xi(p))$. Also, any vector field Y on V lifts to a horizontal vector field Y^* on U by: $Y^*(p) \in H(p)$ and $d_p\xi(Y^*(p)) = Y(\xi(p))$. Note that for any function f and vector field Y on V we have, $Y^*(f^*) = (Y(f))^*$. Also, note that $\pi([Y^*, Z^*]) = [Y, Z]^*$, (again π is the horizontal projection, see Section 2). Indeed, for any function f on V we have,

$$\pi([Y^*, Z^*])(f^*) = [Y^*, Z^*](f^*) = \tag{4.2}$$
$$= Y^*(Z^*(f^*)) - Z^*(Y^*(f^*)) = (Y(Z(f)) - Z(Y(f)))^* = \tag{4.3}$$
$$= ([Y, Z](f))^*. \tag{4.4}$$

Let h be the degenerate metric on the horizon $\mathcal{C}$. Such tensor is the restriction to $\mathcal{C}$ of the spacetime metric g. As $\mathcal{L}_X h = 0$, [6], the metric h on U can be quotient to a metric q on V. Note that $\langle Y^*, Z^* \rangle = \langle Y, Z \rangle^*$, where with some abuse of notation, (that will be used below too), the first bracket corresponds to the degenerate metric h ($\langle Y^*, Z^* \rangle = h(Y^*, Z^*)$) and the second to the metric q ($\langle Y, Z \rangle = q(Y, Z)$).

We show now that the covariant derivative on V defined by,

$$\mathcal{D}_Y Z := d\xi(\pi(\nabla_{Y^*} Z^*)) = d\xi(D_{Y^*} Z^*), \tag{4.5}$$

is indeed the Levi-Civita connection of q (in this formula D is the horizontal covariant derivate on H, see Section 2). Note that though $\pi(\nabla_{Y^*} Z^*)$ is well defined as a vector field on U, it is not necessarily projectable to a vector field on V, so in principle (4.5) may not be well defined. That it is indeed well defined will be clear in the following calculation. By the standard formula, for any vector field W on V we have,

$$\langle \pi(\nabla_{Y^*} Z^*), W^* \rangle = \langle \nabla_{Y^*} Z^*, W^* \rangle = \tag{4.6}$$
$$= \frac{1}{2}\big\{ Z^* \langle Y^*, W^* \rangle + Y^* \langle W^*, Z^* \rangle - W^* \langle Y^*, Z^* \rangle \tag{4.7}$$
$$- \langle [Z^*, W^*], Y^* \rangle - \langle [Y^*, W^*], Z^* \rangle - \langle [Z^*, Y^*], W^* \rangle \big\}. \tag{4.8}$$

Now,

$$Z^* \langle Y^*, W^* \rangle = Z^* \langle Y, W \rangle^* = (Z \langle Y, W \rangle)^*, \tag{4.9}$$

and similarly for the other two terms in (4.7). Also,

$$\langle [Z^*, W^*], Y^* \rangle = \langle \pi([Z^*, W^*]), Y^* \rangle = \langle [Z, W]^*, Y^* \rangle = \langle [Z, W], Y \rangle^*, \tag{4.10}$$

[6] This is because $\mathcal{L}_X h(Y, W) = g(\nabla_Y X, W) + g(\nabla_W X, Y) = g(\omega_X(Y) X, W) + g(\omega_X(W) X, Y) = 0$.

and similarly for the other terms in (4.8). Putting all together we obtain,

$$\langle \pi(\nabla_{Y^*} Z^*), W^* \rangle = \frac{1}{2}\{Z\langle Y, W\rangle + Y\langle W, Z\rangle - W\langle Y, Z\rangle \tag{4.11}$$

$$- \langle [Z, W], Y\rangle - \langle [Y, W], Z\rangle - \langle [Z, Y], W\rangle\}^* = \tag{4.12}$$

$$= \langle \nabla_Y Z, W\rangle^*, \tag{4.13}$$

where on the left hand side of (4.11) the covariant derivative is that of g and on (4.13) the covariant derivative is that of q. Thus $\mathcal{D}$ is the Levi-Civita connection of q.

Let now $\gamma(s)$ be a horizontal curve, namely $\gamma'(s) \in H(\gamma(s))$ for all s. Let $\alpha(s) = \xi(\gamma(s))$. Then, the calculation earlier shows that,

$$d\xi(\pi(\nabla_{\gamma'}\gamma')) = \mathcal{D}_{\alpha'}\alpha'. \tag{4.14}$$

Hence, if $\gamma(s)$ is a horizontal geodesic on U, then $\alpha(s)$ is a geodesic on V. So if $\gamma : (-\epsilon, \epsilon) \to \mathcal{C}$ is a horizontal geodesic on U then $\alpha(s) = \xi(\gamma(s))$ is a geodesic on V with $\alpha(0) = \xi(p)$ and $\alpha'(0) = d\xi(v)$. Conversely, if $\alpha : (-\epsilon, \epsilon) \to V$ is a geodesic on V, with $\alpha(0) = \pi(p)$ and $\alpha'(0) = d\xi(v)$ then one can lift it to a unique horizontal curve $\gamma(s)$, with $\gamma(0) = p$, $\gamma'(0) = v$, that will be the horizontal geodesic we are looking for. □

Now note that $|\gamma'|^{2\prime} = \langle \gamma', \gamma'\rangle' = 2\langle \nabla_{\gamma'}\gamma', \gamma'\rangle = 2\langle \pi(\nabla_{\gamma'}\gamma'), \gamma'\rangle = 0$, and thus the norm of γ' is constant. A standard argument using the compactness of $\mathcal{C}$ then shows that any horizontal geodesic $\gamma : (a, b) \to \mathcal{C}$ can be uniquely continued to a horizontal geodesic $\gamma : (-\infty, \infty) \to \mathcal{C}$, thus proving Proposition 4.

Let us move now to prove Proposition 6. Let us remain for some lines inside the context of the proof just made of Proposition 7. Let p_1 and p_2 be two points in U projecting into the same point, $\xi(p_1) = \xi(p_2)$. Let $v_1 \in H(p_1)$ and $v_2 \in H(p_2)$ projecting into the same vector, $d\xi(v_1) = d\xi(v_2)$. Assume that the norms of v_1 and v_2 (which are necessarily equal) is small enough so that the horizontal geodesics $\gamma_1 : [0, 1] \to \mathcal{C}$ and $\gamma_2 : [0, 1] \to \mathcal{C}$ starting from p_1 and p_2 with velocities v_1 and v_2 respectively lie inside U. Then, $\alpha_1 = \xi(\gamma_1)$ and $\alpha_2 = \xi(\gamma_2)$ are geodesics of V that have the same initial data and are thus equal. This shows that $\xi(\gamma_1(1)) = \xi(\gamma_2(1))$, or, equivalently,

$$\xi(\overline{\exp}(p_2, v_2)) = \xi(\overline{\exp}(p_1, v_1)). \tag{4.15}$$

Now, we claim that v_2 is the horizontal parallel transport of v_1 from $p_1 = \varphi(p_1, 0)$ to $p_2 = \varphi(p_1, z_2)$. Indeed if we let $V(z) \in H(\varphi(p_1, z))$ be the unique horizontal field over the null curve $z \to \varphi(p_1, z)$ such that $d\xi(V(z)) = d\xi(v_1)$, and thus with $V(0) = v_1$ and $V(z_2) = v_2$, then the claim follows by the computation.

$$0 = \pi(\mathcal{L}_X V) = \pi(\nabla_X V - \nabla_V X) = D_X V - \pi(\omega_X(V)X) = D_X V. \tag{4.16}$$

In sum what we have shown is that, given $p \in \mathcal{C}$ and $v \in H(p)$, there are $\epsilon(p) > 0$ and $\delta(p) > 0$ such that, if we let $V(z)$ be the horizontal parallel transport of v along the null curve $z \to \varphi(p, z)$, then the family of horizontal geodesics $\beta(z, s) : [0, \epsilon] \times [-\delta, \delta] \to \mathcal{C}$, given as,

$$\beta(z, s) = \overline{\exp}(\varphi(p, z), sV(z)), \tag{4.17}$$

all project into the same geodesic in V, that is,

$$\xi(\beta(z,s)) = \xi(\beta(0,s)). \tag{4.18}$$

Therefore the curves, $z \to \beta(z,s)$ are all null, and the horizontal fields over them, $z \to \partial_s\beta(z,s)$, are all horizontally parallel. Of course, if ϵ and δ are small enough then $\partial_z\beta(z,s) \neq 0$, for all $z \in [0,\epsilon]$ and $s \in [-\delta,\delta]$. Of course too one can chose $\epsilon(p)$ and $\delta(p)$ such that if p' is sufficiently close to p, then the same holds with $\epsilon(p') = \epsilon(p)$ and $\delta(p') = \delta(p)$.

To prove Proposition 6 we will use what we know so far and make a simple continuity argument. Let $p \in \mathcal{C}$ and $v \in H(p)$, $v \neq 0$ but arbitrary. Let again $V(z)$ be the horizontal parallel transport of v along the null curve $z \to \varphi(p,z)$, for all $z \in \mathbb{R}$. Let $\beta(z,s) = \overline{\exp}(\varphi(p,z), sV(z))$, for all $s \geq 0$. We will show that $\partial_z\beta(z,s)$ is null and different from zero for all $z \in \mathbb{R}$ and all $s \geq 0$. This is clearly enough to prove the proposition. Observe that $z = 0$ doesn't play any particular role, so it is enough to prove that that $\partial_z\beta(0,s)$ is null and different from zero for all $s \geq 0$.

Let s_* be the supremum of the $s_1 > 0$ for which there is $\epsilon = \epsilon(s_1) > 0$ such that for all $s \leq s_1$, we have: (i) the curves $z \to \beta(z,s)$ are null, and $\partial_z\beta(z,s) \neq 0$ for all $z \in [0,\epsilon]$, (ii) the fields $z \to \partial_s\beta(z,s)$ along the curves $z \to \beta(z,s)$, are horizontally parallel. By what was proved earlier we have $s_* > 0$. If $s_* = \infty$ we are done. So assume $0 < s_* < \infty$.

Let $s_1 = s_* - \delta$, for some $\delta > 0$ that we will chose soon. Let $p_1 = \beta(0,s_1)$, $v_1 = \partial_s\beta(0,s_1)$ and let $\beta_1(z,s) = \overline{\exp}(\varphi(p_1,z),(s-s_1)V_1(z))$ where $V_1(z)$ is the horizontal parallel transport of v_1 along $z \to \varphi(p_1,z)$. Then, it is direct that from (i) and (ii) and the fact that $s_1 < s_*$, that there is a function $z_1(z) : [0,\epsilon(s_1)] \to \mathbb{R}$, with $z_1'(z) \neq 0$ for which we have $\beta(z,s) = \beta_1(z_1(z),(s-s_1)V_1(z_1(z)))$, when $s_1 \leq s < s_* + \delta$.

But as shown earlier too, if $\delta > 0$ is sufficiently small, there is $\epsilon(\delta) > 0$, such that: (i') the curves $z \to \beta_1(z,s)$ are null with $\partial_z\beta_1(z,s) \neq 0$ for all $z \in [0,\epsilon(\delta)]$, (ii') the field $z \to \partial_s\beta_1(z,s)$ along the curves $z \to \beta_1(z,s)$, are horizontally parallel.

It follows from the paragraphs above that for $z \in [0,\epsilon(s_1)]$ and for s in the interval $s_1 = s_* - \delta < s < s_* + \delta$ we have $\partial_z\beta_1(z,s) \neq 0$, hence $\partial_z\beta(z,s) = \partial_z\beta_1(z_1(z), s-s_1)z_1'(z)$ is null and different from zero, and, furthermore, $\partial_s\beta(z,s) = \partial_s\beta_1(z_1(z), s-s_1)$ is horizontally parallel. We reach thus a contradiction, having assumed $s_* < \infty$. Thus $s_* = \infty$ and the Proposition 6 is proved.

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